Chapter
Relations and Functions

Topic-1: Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1 MCQs with One Correct Answer

1. If the function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=|x|(x-\sin x)$, then which of the following statements is **TRUE**? **[Adv. 2020]**
(a) f is one-one, but **NOT** onto
(b) f is onto, but **NOT** one-one
(c) f is **BOTH** one-one and onto
(d) f is **NEITHER** one-one **NOR** onto

2. The function f : [0, 3] → [1, 29], defined by $f(x)=2x^3-15x^2+36x+1$, is **[2012]**
(a) one–one and onto (b) onto but not one–one
(c) one–one but not onto (d) neither one–one nor onto

3. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x)=e^{x^2}+e^{-x^2}, g(x)=xe^{x^2}+e^{-x^2}$ and $h(x)=x^2e^{x^2}+e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then **[2010]**
(a) $a=b$ and $c\neq b$ (b) $a=c$ and $a\neq b$
(c) $a\neq b$ and $c\neq b$ (d) $a=b=c$

4. If the functions f(x) and g(x) are defined on R → R such that

$$f(x)=\begin{cases}0, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases};\quad g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$$

then $(f-g)(x)$ is **[2005S]**
(a) one-one & onto
(b) neither one-one nor onto
(c) one-one but not onto
(d) onto but not one-one

5. If $f:[0,\infty)\to[0,\infty)$ and $f(x)=\dfrac{x}{1+x}$, then f is **[2003S]**
(a) one-one and onto (b) one-one but not onto
(c) onto but not one-one (d) neither one-one nor onto

6. Let function $f:R\to R$ be defined by $f(x)=2x+\sin x$ for $x\in R$, then f is **[2002S]**
(a) one-to-one and onto
(b) one-to-one but NOT onto
(c) onto but NOT one-to-one
(d) neither one-to-one nor onto

7. Suppose $f(x)=(x+1)^2$ for $x\geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y=x$, then $g(x)$ equals **[2002S]**
(a) $-\sqrt{x}-1, x\geq 0$ (b) $\dfrac{1}{(x+1)^2}, x>-1$
(c) $\sqrt{x+1}, x\geq -1$ (d) $\sqrt{x}-1, x\geq 0$

8. Let $E=\{1,2,3,4\}$ and $F=\{1,2\}$. Then the number of onto functions from E to F is **[2001S]**
(a) 14 (b) 16 (c) 12 (d) 8

9. The domain of definition of the function $f(x)$ is given by the equation $2^x+2^y=2$ is **[2000S]**
(a) $0<x\leq 1$ (b) $0\leq x\leq 1$
(c) $-\infty<x\leq 0$ (d) $-\infty<x<1$

10. Let $f:R\to R$ be any function. Define $g:R\to R$ by $g(x)=|f(x)|$ for all x. Then g is **[2000S]**
(a) onto if f is onto
(b) one-one if f is one-one
(c) continuous if f is continuous
(d) differentiable if f is differentiable.

11. Let $f(x)$ be defined for all $x>0$ and be continuous. Let $f(x)$ satisfy $f\left(\dfrac{x}{y}\right)=f(x)-f(y)$ for all x, y and $f(e)=1$. Then **[1995S]**
(a) $f(x)$ is bounded (b) $f\left(\dfrac{1}{x}\right)\to 0$ as $x\to 0$
(c) $xf(x)\to 1$ as $x\to 0$ (d) $f(x)=\ln x$

12. The function $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$ where $p > 0, q > 0, r > 0$ assumes its minimum value only on one point if [1995]
(a) $p \neq q$ (b) $r \neq q$
(c) $r \neq p$ (d) $p = q = r$

13. Which of the following functions is periodic? [1983 - 1 Mark]
(a) $f(x) = x - [x]$ where $[x]$ denotes the largest integer less than or equal to the real number x
(b) $f(x) = \sin\frac{1}{x}$ for $x \neq 0$, $f(0) = 0$
(c) $f(x) = x\cos x$
(d) none of these

14. If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$, then [1983 - 1 Mark]
(a) $0 \leq x \leq 4$ (b) $x \leq -2$ or $x \geq 4$
(c) $x \leq 0$ or $x \geq 4$ (d) None of these

15. Let $f(x) = |x-1|$. Then [1983 - 1 Mark]
(a) $f(x^2) = (f(x))^2$ (b) $f(x+y) = f(x) + f(y)$
(c) $f(|x|) = |f(x)|$ (d) None of these

16. The entire graphs of the equation $y = x^2 + kx - x + 9$ is strictly above the x-axis if and only if [1979]
(a) $k < 7$ (b) $-5 < k < 7$
(c) $k > -5$ (d) None of these.

17. Let R be the set of real numbers. If $f: R \to R$ is a function defined by $f(x) = x^2$, then f is : [1979]
(a) Injective but not surjective
(b) Surjective but not injective
(c) Bijective
(d) None of these.

3 Numeric/ New Stem Based Questions

18. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta - \alpha)$ is ______. [Adv. 2018]

4 Fill in the Blanks

19. If f is an even function defined on the interval (-5, 5), then four real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$ are ...., and .... [1996 - 1 Mark]

20. There are exactly two distinct linear functions, ..., and .... which map $[-1, 1]$ onto $[0, 2]$. [1989 - 2 Marks]

21. Let A be a set of n distinct elements. Then the total number of distinct functions from A to A is .... and out of these .... are onto functions. [1985 - 2 Marks]

5 True / False

22. If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively, then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$. [1988 - 1 Mark]

23. The function $f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18}$ is not one -to -one. [1983 - 1 Mark]

24. For real numbers x and y, we write $x * y$ if $x - y + \sqrt{2}$ is an irrational number. Then, the relation* is an equivalence relation. [1981 - 2 Marks]

6 MCQs with One or More than One Correct Answer

25. Let $a \in R$ and let $f: R \to R$ be given by $f(x) = x^5 - 5x + a$. Then [Adv. 2014]
(a) $f(x)$ has three real roots if $a > 4$
(b) $f(x)$ has only real root if $a > 4$
(c) $f(x)$ has three real roots if $a < -4$
(d) $f(x)$ has three real roots if $-4 < a < 4$

26. The function $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$ has a local minimum or a local maximum at $x =$ [Adv. 2013]
(a) -2 (b) $\frac{-2}{3}$ (c) 2 (d) $\frac{2}{3}$

27. Let $f: (-1, 1) \to IR$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value (s) of $f\left(\frac{1}{3}\right)$ is (are) [Adv. 2012]
(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$ (c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$

28. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where $[x]$ stands for the greatest integer function, then [1991 - 2 Marks]
(a) $f\left(\frac{\pi}{2}\right) = -1$ (b) $f(\pi) = 1$
(c) $f(-\pi) = 0$ (d) $f\left(\frac{\pi}{4}\right) = 1$

29. Let $g(x)$ be a function defined on $[-1, 1]$. If the area of the equilateral triangle with two of its vertices at $(0,0)$ and $[x, g(x)]$ is $\frac{\sqrt{3}}{4}$, then the function $g(x)$ is [1989 - 2 Marks]
(a) $g(x) = \pm\sqrt{1-x^2}$ (b) $g(x) = \sqrt{1-x^2}$
(c) $g(x) = -\sqrt{1-x^2}$ (d) $g(x) = \sqrt{1+x^2}$

30. If $y = f(x) = \frac{x+2}{x-1}$ then [1984 - 3 Marks]
(a) $x = f(y)$ (b) $f(1) = 3$
(c) y increases with x for $x < 1$
(d) f is a rational function of x

7 Match the Following

31. Match the statements given in Column-I with the intervals/union of intervals given in Column-II. **[2011]**

Column-I	Column-II
(A) The set $\left\{\text{Re}\left(\dfrac{2iz}{1-z^2}\right):\right.$ z is a complex number, $\left.\|z\| = 1\ z \neq \pm 1\right\}$ is	(p) $(-\infty,-1)\cup(1,\infty)$
(B) The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty,-0)\cup(0,\infty)$
(C) If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \dfrac{\pi}{2}\right\}$ is	(r) $[2,\infty)$
	(s) $(-\infty,-1]\cup[1,\infty)$
(D) If $f(x) = x^{3/2}(3x-10), x \ge 0$ then $f(x)$ is increasing in	(t) $(-\infty,0]\cup[2,\infty)$

32. Let $f(x) = \dfrac{x^2-6x+5}{x^2-5x+6}$

Match of expressions/statements in Column I with expressions/statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS. **[2007 -6 marks]**

Column I	Column II
(A) If $-1 < x < 1$, then $f(x)$ satisfies	(p) $0 < f(x) < 1$
(B) If $1 < x < 2$, then $f(x)$ satisfies	(q) $f(x) < 0$
(C) If $3 < x < 5$, then $f(x)$ satisfies	(r) $f(x) > 0$
(D) If $x > 5$, then $f(x)$ satisfies	(s) $f(x) < 1$

33. Let the function defined in column 1 have domain $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and range $(-\infty, \infty)$ **[1992 - 2 Marks]**

Column I	Column II
(A) $1+2x$	(p) onto but not one-one
(B) $\tan x$	(q) one- one but not onto
	(r) one- one and onto
	(s) neither one-one nor onto

8 Comprehension/Passage Based Questions

PARAGRAPH "I"

Let S = {1, 2, 3, 4, 5, 6} and X be the set of all relations R from S to S that satisfy both the following properties:

(i) R has exactly 6 elements.

(ii) For each (a, b) ∈ R, we have $|a-b| \ge 2$.

Let Y = {R ∈ X : The range of R has exactly one element} and

Z = {R ∈ X : R is a function from S to S}.

Let $n(A)$ denote the number of elements in a set A.

34. If $n(X) = {}^{m}C_6$, then the value of m is __________. **[Adv. 2024]**

35. If the value of $n(Y) + n(Z)$ is k^2, then $|k|$ is _______. **[Adv. 2024]**

10 Subjective Problems

36. Let $f(x) = Ax^2 + Bx + C$ where A, B, C are real numbers. Prove that if $f(x)$ is an integer whenever x is an integer, then the numbers $2A$, $A+B$ and C are all integers. Conversely, prove that if the numbers $2A$, $A+B$ and C are all integers then $f(x)$ is an integer whenever x is an integer. **[1998 - 8 Marks]**

37. A function $f : IR \to IR$, where IR is the set of real numbers, is defined by $f(x) = \dfrac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$. Find the interval of values of α for which f is onto. Is the function one-to-one for $\alpha = 3$? Justify your answer. **[1996 - 5 Marks]**

38. Let $\{x\}$ and $[x]$ denotes the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$. **[1994 - 4 Marks]**

39. Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function '$f$' satisfies the relation $f(x+y) = f(x)f(y)$ for all natural numbers x, y and further $f(1) = 2$. **[1992 - 6 Marks]**

40. Let R be the set of real numbers and $f : R \to R$ be such that for all x and y in R $|f(x) - f(y)| \le |x-y|^3$. Prove that $f(x)$ is a constant. **[1988 - 2 Marks]**

41. A relation R on the set of complex numbers is defined by $z_1 R z_2$ if and only if $\dfrac{z_1 - z_2}{z_1 + z_2}$ is real. Show that R is an equivalence relation. **[1982 - 2 Marks]**

42. Let A and B be two sets each with a finite number of elements. Assume that there is an injective mapping from A to B and that there is an injective mapping from B to A. Prove that there is a bijective mapping from A to B. **[1981 - 2 Marks]**

43. Consider the following relations in the set of real numbers R.

$R = \{(x, y); x \in R, y \in R, x^2 + y^2 \le 25\}$

$R' = \left\{(x, y) : x \in R, y \in R, y \ge \frac{4}{9}x^2\right\}$

Find the domain and range of $R \cap R'$. Is the relation $R \cap R'$ a function? [1979]

44. If $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$, find $f(6)$. [1979]

45. Draw the graph of $y = |x|^{1/2}$ for $-1 \le x \le 1$. [1978]

46. Find the domain and range of the function $f(x) = \frac{x^2}{1 + x^2}$. Is the function one-to-one? [1978]

Topic-2: Composite Functions & Relations, Inverse of a Function, Binary Operations

1 MCQs with One Correct Answer

1. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is [Adv. 2011]

(a) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$

(b) $\pm\sqrt{n\pi}, n \in \{1, 2, ...\}$

(c) $\frac{\pi}{2} + 2n\pi, n \in \{.... -2, -1, 0, 1, 2 ....\}$

(d) $2n\pi, n \in \{... -2, -1, 0, 1, 2, ...\}$

2. X and Y are two sets and $f : X \to Y$. If $\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x; d \subset Y, x \subset X\}$, then the true statement is [2005S]

(a) $f(f^{-1}(b)) = b$ (b) $f^{-1}(f(a)) = a$

(c) $f(f^{-1}(b)) = b, b \subset y$ (d) $f^{-1}(f(a)) = a, a \subset x$

3. If $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$, then $g(f(x))$ is invertible in the domain [2004S]

(a) $\left[0, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$

4. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ such that $\min f(x) > \max g(x)$, then the relation between b and c, is [2003S]

(a) no real value of b & c (b) $0 < c < b\sqrt{2}$

(c) $|c| < |b|\sqrt{2}$ (d) $|c| > |b|\sqrt{2}$

5. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued x, is [2003S]

(a) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(c) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (d) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

6. Let $f(x) = \frac{\alpha x}{x + 1}, x \ne -1$. Then, for what value of α is $f(f(x)) = x$? [2001S]

(a) $\sqrt{2}$ (b) $-\sqrt{2}$

(c) 1 (d) -1

7. The domain of definition of $f(x) = \frac{\log_2(x + 3)}{x^2 + 3x + 2}$ is [2001S]

(a) $R \setminus \{-1, -2\}$ (b) $(-2, \infty)$

(c) $R \setminus \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

8. If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$ then $f^{-1}(x)$ equals

(a) $(x + \sqrt{x^2 - 4})/2$ (b) $x/(1 + x^2)$ [2001S]

(c) $(x - \sqrt{x^2 - 4})/2$ (d) $1 + \sqrt{x^2 - 4}$

9. Let $g(x) = 1 + x - [x]$ and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to [2001S]

(a) x (b) 1

(c) $f(x)$ (d) $g(x)$

10. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is [1999 - 2 Marks]

(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1+\sqrt{1+4\log_2 x})$

(c) $\frac{1}{2}(1-\sqrt{1+4\log_2 x})$ (d) not defined

11. Let $f(x)=(x+1)^2-1, x\ge -1$. Then the set $\{x: f(x)=f^{-1}(x)\}$ is [1995]

(a) $\left\{0,-1,\frac{-3+i\sqrt{3}}{2},\frac{-3-i\sqrt{3}}{2}\right\}$

(b) $\{0, 1, -1\}$

(c) $\{0, -1\}$

(d) empty

12. Let $f(x)=\sin x$ and $g(x)=\ln|x|$. If the ranges of the composition functions fog and gof are R_1 and R_2 respectively, then [1994 - 2 Marks]

(a) $R_1=\{u: -1\le u<1\}, R_2=\{v: -\infty<v<0\}$

(b) $R_1=\{u: -\infty<u<0\}, R_2=\{v: -1\le v\le 0\}$

(c) $R_1=\{u: -1<u<1\}, R_2=\{v: -\infty<v<0\}$

(d) $R_1=\{u: -1\le u\le 1\}, R_2=\{v: -\infty<v\le 0\}$

13. The domain of definition of the function $y=\frac{1}{\log_{10}(1-x)}+\sqrt{x+2}$ is [1983 - 1 Mark]

(a) $(-3,-2)$ excluding -2.5 (b) $[0, 1]$ excluding 0.5

(c) $[-2, 1)$ excluding 0 (d) none of these

14. If $f(x)=\cos(\ln x)$, then $f(x)f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$ has the value [1983 - 1 Mark]

(a) -1 (b) $1/2$

(c) -2 (d) none of these

2 Integer Value Answer/ Non-Negative Integer

15. The value of $((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}}\times(\sqrt{7})^{\frac{1}{\log_4 7}}$ is __________. [Adv. 2018]

16. Let $f: [0, 4\pi]\to[0,\pi]$ be defined by $f(x)=\cos^{-1}(\cos x)$. The number of points $x\in[0,4\pi]$ satisfying the equation $g(x)=\frac{10-x}{10}$ is [Adv. 2014]

4 Fill in the Blanks

17. If $f(x)=\sin^2 x+\sin^2\left(x+\frac{\pi}{3}\right)+\cos x\cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right)=1$, then $(gof)(x)=$ .... [1996 - 2 Marks]

18. If $f(x)=\sin\ln\left(\frac{\sqrt{4-x^2}}{1-x}\right)$, then domain of $f(x)$ is .... and its range is .... [1985 - 2 Marks]

19. The domain of the function $f(x)=\sin^{-1}(\log_2\frac{x^2}{2})$ is given by .... [1984 - 2 Marks]

20. The values of $f(x)=3\sin\left(\sqrt{\frac{\pi^2}{16}-x^2}\right)$ lie in the interval .... [1983 - 1 Mark]

5 True / False

21. If $f(x)=(a-x^n)^{1/n}$ where $a>0$ and n is a positive integer, then $f[f(x)]=x$. [1983 - 1 Mark]

6 MCQs with One or More than One Correct Answer

22. Let $f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x\in R$ and $g(x)=\frac{\pi}{2}\sin x$ for all $x\in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true? [Adv. 2015]

(a) Range of f is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(b) Range of fog is $\left[-\frac{1}{2},\frac{1}{2}\right]$

(c) $\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{\pi}{6}$

(d) There is an $x\in R$ such that $(gof)(x)=1$

23. Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ be given by $f(x)=(\log(\sec x+\tan x))^3$. Then [Adv. 2014]

(a) $f(x)$ is an odd function

(b) $f(x)$ is one-one function

(c) $f(x)$ is an onto function

(d) $f(x)$ is an even function

24. Let $f: (0, 1)\to \mathbf{R}$ be defined by $f(x)=\frac{b-x}{1-bx}$, where b is a constant such that $0<b<1$. Then [2011]

(a) f is not invertible on $(0, 1)$

(b) $f\ne f^{-1}$ on $(0, 1)$ and $f'(b)=\frac{1}{f'(0)}$

(c) $f=f^{-1}$ on $(0, 1)$ and $f'(b)=\frac{1}{f'(0)}$

(d) f^{-1} is differentiable $(0, 1)$

25. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then [1998 - 2 Marks]

(a) $f(x) = \sin^2 x,\ g(x) = \sqrt{x}$

(b) $f(x) = \sin x,\ g(x) = |x|$

(c) $f(x) = x^2,\ g(x) = \sin\sqrt{x}$

(d) f and g cannot be determined.

26. If $f(x) = 3x - 5$, then $f^{-1}(x)$ [1998 - 2 Marks]

(a) is given by $\dfrac{1}{3x-5}$

(b) is given by $\dfrac{x+5}{3}$

(c) does not exist because f is not one-one

(d) does not exist because f is not onto.

7 Match the Following

27. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \dfrac{x}{x-1} > 0\right\}$ and

$E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

(Here, the inverse trigonometric function $\sin^{-1} x$ assumes values in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$).

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\dfrac{x}{x-1}\right)$ and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right)$. [Adv. 2018]

LIST - I	LIST - II
P. The range of f is	1. $\left(-\infty, \dfrac{1}{1-e}\right] \cup \left[\dfrac{e}{e-1}, \infty\right)$
Q. The range of g contains	2. $(0, 1)$
R. The domain of f contains	3. $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$
S. The domain of g is	4. $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \dfrac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\dfrac{1}{2}, \dfrac{e}{e-1}\right]$

The correct option is:

(a) P → 4; Q → 2; R → 1; S → 1

(b) P → 3; Q → 3; R → 6; S → 5

(c) P → 4; Q → 2; R → 1; S → 6

(d) P → 4; Q → 3; R → 6; S → 5

10 Subjective Problem

28. Let f be a one-one function with domain $\{x, y, z\}$ and range $\{1, 2, 3\}$. It is given that exactly one of the following statements is true and the remaining two are false $f(x) = 1,\ f(y) \neq 1,\ f(z) \neq 2$ determine $f^{-1}(1)$. [1982 - 3 Marks]

Answer Key

Topic-1 : Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1. (c) 2. (b) 3. (d) 4. (a) 5. (b) 6. (a) 7. (d) 8. (a) 9. (d) 10. (c)
11. (d) 12. (c) 13. (a) 14. (c) 15. (d) 16. (b) 17. (d) 18. (119) 19. $\dfrac{3 \pm \sqrt{5}}{2}, \dfrac{-3 \pm \sqrt{5}}{2}$
20. $x + 1, -x + 1$ 21. $n^n, n!$ 22. (False) 23. (True) 24. (False) 25. (b, d) 26. (a, b) 27. (a, b) 28. (a, c)
29. (b, c) 30. (a, d) 31. (A) → (s), (B) → (t), (C) → (r), (D) → (r)
32. (A) → (r), (s), (p); (B) → (q), (s); (C) → (q), (s); (D) → (r), (s), (p) 33. (A) → (q); (B) → (r)
34. (20) 35. (36)

Topic-2 : Composite Functions & Relations, Inverse of a Function, Binary Operations

1. (a) 2. (d) 3. (b) 4. (d) 5. (a) 6. (d) 7. (d) 8. (a) 9. (b) 10. (b)
11. (c) 12. (d) 13. (c) 14. (d) 15. (8)0 16. (3) 17. (1) 18. (–2, 1) [–1, 1]
19. $[-2, -1] \cup [1, 2]$ 20. $[0, 3/\sqrt{2}]$ 21. (True) 22. (a, b, c) 23. (a, b, c)
24. (a, b) 25. (a) 26. (b) 27. (a)

Hints & Solutions

Topic-1: Types of Relations, Inverse of a Relation, Mappings, Mapping of Functions, Kinds of Mapping of Functions

1. **(c)** $f(x)$ is a non-periodic, continous and odd function

$$f(x)=\begin{cases}-x^2+x\sin x, x<0\\ x^2-x\sin x, x\ge 0\end{cases}$$

$$f'(x)=\begin{cases}-2x+\sin x+x\cos x,\ x<0\\ 2x-\sin x-x\cos x,\ x\ge 0\end{cases}$$

$$f'(x)=\begin{cases}-(x-\sin x)-x(1-\cos x), x<0\\ (x-\sin x)+x(1-\cos x), x>0\end{cases}$$

$\because$ $x-\sin x<0$ if $x<0$ and $1-\cos x>0,\ \forall\ x\in R$

$\therefore$ $-(x-\sin x)-x(1-\cos x)>0$ if $x<0$

and $(x-\sin x)+x(1-\cos x)>0$ if $x>0$

$\Rightarrow$ $f'(x)>0\ \forall\ x\in R \Rightarrow$ $f(x)$ is increasing in R

$\Rightarrow$ $f(x)$ is one-one

$\therefore\ \underset{x\to-\infty}{Lt}\left(-x^2\right)\left(1-\dfrac{\sin x}{x}\right)=-\infty \therefore\ \underset{x\to\infty}{Lt}\ x^2\left(1-\dfrac{\sin x}{x}\right)=\infty$

$\Rightarrow$ Range of $f(x)=R$ $\Rightarrow$ $f(x)$ is an onto function

2. **(b)** Given : $f(x)=2x^3-15x^2+36x+1$

$\Rightarrow$ $f'(x)=6x^2-30x+36$
$=6(x^2-5x+6)=6(x-2)(x-3)$

$\because$ $f'(x)>0\ \ \forall\ x\in[0,2)$ and $f'(x)<0\ \ \forall\ x\in(2,3)$

$\therefore$ $f(x)$ is increasing on $[0,\ 2)$ and decreasing on $(2,3)$

$\therefore$ $f(x)$ is many one on $[0,3]$

Also $f(0)=1, f(2)=29, f(3)=28$

$\therefore$ Absolute min $=1$ and Absolute max $=29$

$\therefore$ Range of $f=[1,29]=$ codomain

Hence f is onto.

3. **(d)** $f(x)=e^{x^2}+e^{-x^2}\Rightarrow f'(x)=2x\left(e^{x^2}-e^{-x^2}\right)\ge 0$, $\forall\ x\in[0,1]$

$\therefore$ f (x) is an increasing function on [0, 1]

$\therefore$ $f_{max}=f(1)=e+\dfrac{1}{e}=a$; $g(x)=xe^{x^2}+e^{-x^2}$

$\Rightarrow$ $g'(x)=(2x^2+1)e^{x^2}-2xe^{-x^2}\ge 0,\forall x\in[0,1]$

$\therefore$ $g(x)$ is an increasing function on [0, 1]

$\therefore$ $g_{\max}=g(1)=e+\dfrac{1}{e}=b$

$h(x)=x^2e^{x^2}+e^{-x^2}$

$\Rightarrow$ $h'(x)=2x\left[e^{x^2}(1+x^2)-e^{-x^2}\right]\ge 0,\forall x\in[0,1]$

$\therefore$ $h(x)$ is an increasing function on $[0,1]$

$\therefore$ $h_{\max}=h(1)=e+\dfrac{1}{e}=c$ $\therefore a=b=c.$

4. **(a)** Given $f(x)$ and $g(x)$ defined on $R\to R$

and $f(x)=\begin{cases}0, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases}$

$g(x)=\begin{cases}0, & x\in\text{irrational}\\ x, & x\in\text{rational}\end{cases}$

$\therefore (f-g):R\to R$ such that

$(f-g)(x)=\begin{cases}-x, & x\in\text{rational}\\ x, & x\in\text{irrational}\end{cases}$

Since $(f-g):R\to R$ for any x, then there is only one value of $(f(x)-g(x))$ whether x is rational or irrational. Moreover as $x\in R, f(x)-g(x)$ also belongs to R. Therefore, $(f-g)$ is one-one onto.

5. **(b)** Given : $f:[0,\infty)\to[0,\infty)$ and $f(x)=\dfrac{x}{x+1}$

$\therefore$ $f'(x)=\dfrac{1+x-x}{(1+x)^2}=\dfrac{1}{(1+x)^2}>0\,\forall\,x$

$\therefore$ f is an increasing function $\Rightarrow f$ is one-one.

Now, $D_f=[0,\infty)$

For range let $\dfrac{x}{1+x}=y\Rightarrow x=\dfrac{y}{1-y}$

Now, $x\ge 0\Rightarrow 0\le y<1$

$\therefore R_f=[0,1)\ne$ Co-domain, $\therefore$ f is not onto.

6. **(a)** Given : $f(x)=2x+\sin x,\ x\in R$

$\Rightarrow f'(x)=2+\cos x$. Now $-1\le\cos x\le 1$

$\Rightarrow 1\le 2+\cos x\le 3\Rightarrow 1\le 2+\cos x\le 3$

$\therefore f'(x)>0,\ \forall\,x\in R$

$\Rightarrow f(x)$ is strictly increasing and therefore one-one

Also as $x\to\infty$, $f(x)\to\infty$ and $x\to-\infty$, $f(x)\to-\infty$

$\therefore$ Range of $f(x)=R=$ domain of $f(x)\Rightarrow f(x)$ is onto.

Hence, $f(x)$ is one-one and onto.

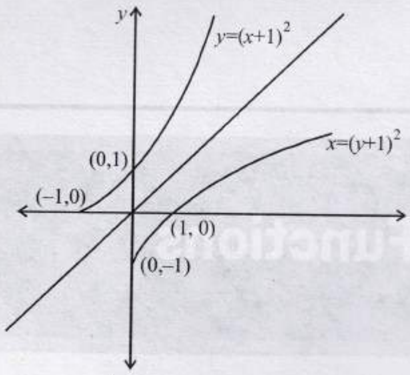
7. **(d)** Given : $f(x)=(x+1)^2, x\ge -1$

If $g(x)$ is the reflection of $f(x)$ in the line $y=x$, then it can be obtained by interchanging x and y in $f(x)$

i.e., $y=(x+1)^2$ changes to $x=(y+1)^2$

$\Rightarrow$ $y+1=\sqrt{x}$ $[y+1\ne-\sqrt{x}$, since $y\ge -1]$

$\Rightarrow$ $y=\sqrt{x}-1$ defined $\forall\ x\ge 0$



$\therefore g(x) = \sqrt{x} - 1 \quad \forall\ x \geq 0$

8. **(a)** $E = \{1, 2, 3, 4\}$ and $F = \{1, 2\}$

From E to F we can define, in all, $2 \times 2 \times 2 \times 2 = 16$ functions (2 options for each element of E) out of which 2 are into, when all the elements of E either map to 1 or to 2.

$\therefore$ Number of onto functions $= 16 - 2 = 14$

9. **(d)** Given : $2^x + 2^y = 2\ \forall\ x, y \in R$

but $2^x, 2^y > 0\ \forall\ x, y \in R$

$\therefore \quad 2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2 \Rightarrow x < 1$

Hence domain $= (-\infty, 1)$

10. **(c)** Let $h(x) = |x|$

$\therefore \quad g(x) = |f(x)| = h(f(x))$

Since composition of two continuous functions is continuous, therefore g is continuous if f is continuous.

11. **(d)** $f(x)$ is continuous and defined for all $x > 0$.

Also $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $f(e) = 1$

$\Rightarrow$ Clearly $f(x) = \ell n\, x$, satisfies all these properties

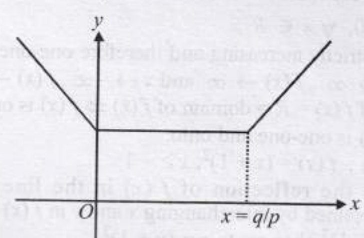
$\therefore f(x) = \ell n\, x$

12. **(c)** $f(x) = |px - q| + r|x|$

$$= \begin{cases} -px + q - rx, & x \leq 0 \\ -px + q + rx, & 0 < x \leq q/p \\ px - q + rx, & q/p < x \end{cases}$$

$$f'(x) = \begin{cases} -p - r, & x \leq 0 \\ -p + r, & 0 < x \leq q/p \\ p + r, & q/p < x \end{cases}$$

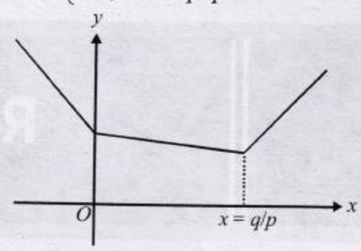
For $r = p$, $f'(x) \begin{cases} < 0, \text{ if } x < 0 \\ = 0, \text{ if } 0 < x \leq q/p \\ > 0, \text{ if } > q/p \end{cases}$



(i)

From graph (i), infinite many points for minima value of $f(x)$

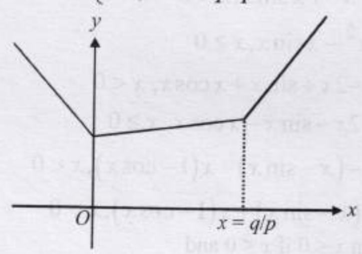
For $r < p$, $f'(x) \begin{cases} < 0, \text{ if } x \leq 0 \\ < 0, \text{ if } 0 < x \leq q/p \\ > 0, \text{ if } x < q/p \end{cases}$



(ii)

From graph (ii), only point of minima of $f(x)$ at $x = q/p$

For $r > p$, $f'(x) \begin{cases} < 0, \text{ if } x \leq 0 \\ > 0, \text{ if } 0 < x \leq q/p \\ > 0, \text{ if } x > q/p \end{cases}$

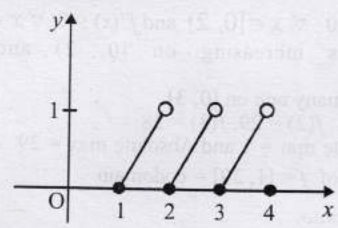


(iii)

From graph (iii), only one point of minima of $f(x)$ at $x = 0$

13. **(a)** $f(x) = x - [x] = \begin{cases} x - 1, & 1 \leq x < 2 \\ x - 2, & 2 \leq x < 3 \\ x - 3, & 3 \leq x < 4 \\ \ldots \end{cases}$

$\therefore$ Graph of function $f(x)$ is

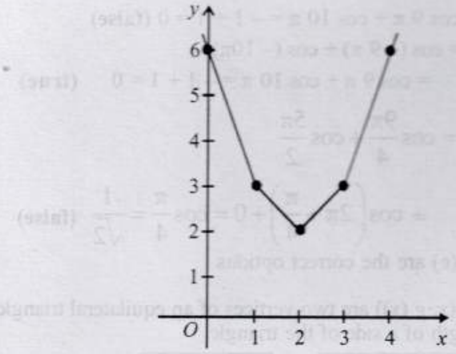


Clearly it is a periodic function with period 1.

14. **(c)** $|x-1| + |x-2| + |x-3| \geq 6$

Consider $f(x) = |x-1| + |x-2| + |x-3|$

$$\therefore f(x) = \begin{cases} 6 - 3x, & x < 1 \\ 4 - x, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \\ 3x - 6, & x \geq 3 \end{cases}$$



From the graph of $f(x)$, it is clear that $f(x) \geq 6$ for $x \leq 0$ or $x \geq 4$

15. **(d)** Given : $f(x) = |x-1| = \begin{cases} -x+1, & x<1 \\ x-1, & x \geq 1 \end{cases}$

Consider $f(x^2) = (f(x))^2$

If it is true, it should be true for all x.

Put $x = 2$, then

LHS $= f(2^2) = |4-1| = 3$ and RHS $= (f(2))^2 = 1$

Since, L.H.S. $\neq$ R.H.S.

$\therefore$ (a) is not correct.

Consider $f(x+y) = f(x) + f(y)$

Put $x = 2, y = 5$, then

L.H.S. $= f(7) = 6$ and R.H.S $= f(2) + f(5) = 1 + 4 = 5$

$\therefore$ (b) is not correct.

Consider $f(|x|) = |f(x)|$

Put $x = -5$, then L.H.S. $= f(|-5|) = f(5) = 4$

and R.H.S. $= |f(-5)| = |-5-1| = 6$

$\therefore$ (c) is not correct.

$\therefore$ (d) is the correct alternative.

16. **(b)** $y = x^2 + (k-1)x + 9 = \left(x + \frac{k+1}{2}\right)^2 + 9 - \left(\frac{k-1}{2}\right)^2$

For entire graph to be above x-axis, we should have

$9 - \left(\frac{k-1}{2}\right)^2 > 0$

$$\underset{-5}{\circ} \quad - \quad \underset{7}{\circ} \quad (+ \text{ outside})$$

$\Rightarrow \quad k^2 - 2k - 35 < 0 \Rightarrow (k-7)(k+5) < 0$

$\Rightarrow \quad -5 < k < 7$

17. **(d)** $f(x) = x^2$ is many one as $f(1) = f(-1) = 1$

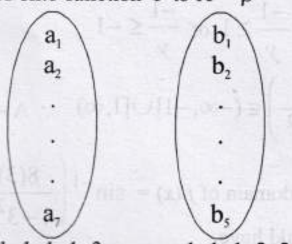
Also f is into as – ve real number have no pre-image.

$\therefore$ F is neither injective nor surjective.

18. **(119)** Here n (X) = 5 and n (Y) = 7

Number of one-one function $= \alpha = {}^7C_5 \times 5!$

and Number of onto function Y to X = β



1, 1, 1, 1, 3   1, 1, 1, 2, 2

$= \frac{7!}{3!\,4!} \times 5! + \frac{7!}{(2!)^3\,3!} \times 5! = ({}^7C_3 + 3 \times {}^7C_3)\,5!$

$= 4 \times {}^7C_3 \times 5!$

$\Rightarrow \quad \frac{\beta - \alpha}{5!} = 4 \times {}^7C_3 - {}^7C_5 = 4 \times 35 - 21 = 119$

19. Given an even function $f(x) = f\left(\frac{x+1}{x+2}\right)$

$\therefore \quad f(x) = f(-x) = f\left(\frac{-x+1}{-x+2}\right)$

$\Rightarrow \quad x = \frac{-x+1}{-x+2} \Rightarrow x^2 - 3x + 1 = 0 \quad \Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$

Also $f(x) = f\left(\frac{x+1}{x+2}\right) = f(-x)$

$\Rightarrow \quad \frac{x+1}{x+2} = -x \Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$

$\therefore$ Four values of x are

$\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}$ and $\frac{-3-\sqrt{5}}{2}$

20. Every linear function is either strictly increasing or strictly decreasing. If $f(x) = ax + b$, $D_f = [p, q]$, $R_f = [m, n]$.

Then $f(p) = m$ and $f(q) = n$, if $f(x)$ is strictly increasing and $f(p) = n, f(q) = m$, if $f(x)$ is strictly decreasing function.

Let the linear function $f(x) = ax + b$, maps $[-1, 1]$ onto $[0, 2]$.

Then $f(-1) = 0$ and $f(1) = 2$ or $f(-1) = 2$ and $f(1) = 0$, depending upon $f(x)$ is increasing or decreasing respectively.

$\Rightarrow \quad -a + b = 0$ and $a + b = 2$ ....(i)

or $\quad -a + b = 2$ and $a + b = 0$ ....(ii)

On solving (i), we get $a = 1, b = 1$.

On solving (ii), we get $a = -1, b = 1$

Hence, there are only two functions $f(x) = x + 1$ and $f(x) = -x + 1$.

21. Set A has n distinct elements.

Then to define a function from A to A, we need to associate each element of set A to any one the n elements of set A.

$\therefore \quad$ Total number of functions from A to $A = n^n$

Now for an onto function from A to A, we need to associate each element of A to one and only one element of A.

$\therefore \quad$ Total number of functions from A to $A = n!$.

22. **(False)** We know that sum of any two functions is defined only on the points where both f_1 as well as f_2 are defined that is $f_1 + f_2$ is defined on $D_1 \cap D_2$.

$\therefore$ The given statement is false.

23. **(True)** A function is one-one if it is strictly increasing or strictly decreasing, other wise it is many one.

$f(x) = \frac{x^2 + 4x + 30}{x^2 - 8x + 18} \Rightarrow f'(x) = \frac{-12[x^2 + 2x - 26]}{(x^2 - 8x + 18)^2}$

$\Rightarrow \quad f'(x) = \frac{-12(x - 3\sqrt{3} + 1)(x + 3\sqrt{3} + 1)}{(x^2 - 8x + 18)^2}$

$\Rightarrow \quad f(x)$ increases on $\left(-3\sqrt{3} - 1, 3\sqrt{3} - 1\right)$ and decreases otherwise.

$\therefore \quad f(x)$ is many one.

24. **(False)** Given: $x * y = x - y + \sqrt{2}$

Let $x = 2\sqrt{2}$, $y = \sqrt{2}$

$\Rightarrow x * y = 2\sqrt{2} - \sqrt{2} + \sqrt{2} = 3\sqrt{2}$ (irrational)

and $y * x = \sqrt{2} - 2\sqrt{2} + \sqrt{2} = 0$ (rational)

$\therefore x * y \neq y * x$ (Not symmetric)

Hence $*$ is not an equivalence relation.

25. **(b, d)** $f(x) = x^5 - 5x + a$

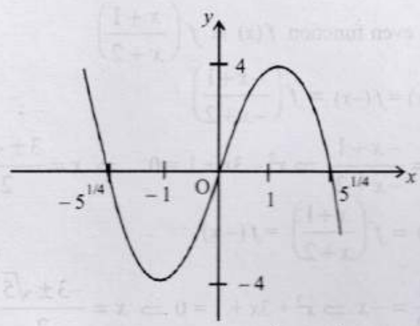
$f(x) = 0 \Rightarrow x^5 - 5x + a = 0 \Rightarrow a = 5x - x^5 = g(x)$

$\Rightarrow g(x) = 0$ when $x = 0, 5^{1/4}, -5^{1/4}$

and $g'(x) = 0 \Rightarrow x = 1, -1$

Also $g(-1) = -4$ and $g(1) = 4$

Thus graph of $g(x)$ will be as shown below.

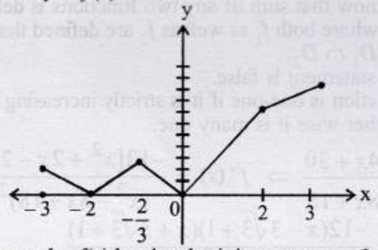


From graph, it is clear that if $a \in (-4, 4)$
then $g(x) = a$ or $f(x) = 0$ has 3 real roots
If $a > 4$ or $a < -4$
then $f(x) = 0$ has only one real root.
∴ option (b) and (d) are the correct options.

26. **(a, b)** Given : $f(x) = 2|x| + |x+2| - \big||x+2| - 2|x|\big|$
Critical points of the $f(x)$ can be obtained by solving $|x| = 0$, $|x+2| = 0$ and $\big||x+2| - 2|x|\big| = 0$, which give
$x = 0, -2, 2, -\dfrac{2}{3}$

$$\therefore \quad f(x) \begin{cases} -2x-4, & x \le -2 \\ 2x+4, & -2 < x \le -\dfrac{2}{3} \\ -4x, & -\dfrac{2}{3} < x \le 0 \\ 4x, & 0 < x \le 2 \\ 2x+4, & x > 2 \end{cases}$$

Graph of $y = f(x)$ is as follows :



From graph, $f(x)$ has local minimum at $x = -2$ and $x = 0$ and local maximum at $x = -\dfrac{2}{3}$

27. **(a, b)** Given : $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta} = \dfrac{2\cos^2\theta}{2\cos^2\theta - 1}$

$$= \frac{1 + \cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$$

Let $\cos 4\theta = \dfrac{1}{3} \Rightarrow 2\cos^2 2\theta - 1 = \dfrac{1}{3} \Rightarrow \cos 2\theta = \pm\sqrt{\dfrac{2}{3}}$

$\therefore f(\cos 4\theta) = 1 + \dfrac{1}{\cos 2\theta} = 1 \pm \sqrt{\dfrac{3}{2}}$ or $f\left(\dfrac{1}{3}\right) = 1 \pm \sqrt{\dfrac{3}{2}}$

28. **(a, c)** $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$
We know that $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$
$\Rightarrow [\pi^2] = 9$ and $[-\pi^2] = -10$
∴ $f(x) = \cos 9x + \cos(-10x)$
$f(x) = \cos 9x + \cos 10x$

(a) $f\left(\dfrac{\pi}{2}\right) = \cos\dfrac{9\pi}{2} + \cos 5\pi = -1$ **(true)**

(b) $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ **(false)**

(c) $f(-\pi) = \cos(-9\pi) + \cos(-10\pi)$
$= \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ **(true)**

(d) $f\left(\dfrac{\pi}{4}\right) = \cos\dfrac{9\pi}{4} + \cos\dfrac{5\pi}{2}$
$= \cos\left(2\pi + \dfrac{\pi}{4}\right) + 0 = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ **(false)**

∴ (a) and (c) are the correct options.

29. **(b, c)**
As (0, 0) and $(x, g(x))$ are two vertices of an equilateral triangle; therefore, length of a side of the triangle
$= \sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$

∴ The area of equilateral triangle $= \dfrac{\sqrt{3}}{4}(x^2 + (g(x))^2)$

But given that area of the equilateral triangle $= \dfrac{\sqrt{3}}{4}$

∴ $(g(x))^2 = 1 - x^2 \Rightarrow g(x) = \pm\sqrt{1 - x^2}$
∴ (b), (c) are the correct options as (a) is not a function.
(∵ image of x is not unique)

30. **(a, d)** Given : $f(x) = y = \dfrac{x+2}{x-1}$

(a) $f(x) = \dfrac{x+2}{x-1} = y \Rightarrow x = f(y)$
∴ (a) is correct

(b) $f(1) \ne 3$ as function is not defined for $x = 1$
∴ (b) is not correct.

(c) $f'(x) = \dfrac{x-1-x-2}{(x-1)^2} = \dfrac{-3}{(x-1)^2}$

∴ $f'(x) < 0$, if $x \ne 1 \Rightarrow f(x)$ is decreasing if $x \ne 1$
∴ (c) is not correct.

(d) $f(x) = \dfrac{x+2}{x-1}$, which is a rational function of x.
∴ (d) is correct.

31. **(A) → (s), (B) → (t), (C) → (r), (D) → (r)**
Let $z = x + iy$. Given that $|z| = 1$ i.e. $x^2 + y^2 = 1$ and $x \ne \pm 1$

Then $\text{Re}\left(\dfrac{2iz}{1 - z^2}\right) = \text{Re}\left(\dfrac{2iz}{z.\bar{z} - z^2}\right)$

$= \text{Re}\left(\dfrac{2i}{\bar{z} - z}\right) = \text{Re}\left(\dfrac{2i}{-2iy}\right) = \text{Re}\left(\dfrac{-1}{y}\right) = \dfrac{-1}{y}$

where, $x = \sqrt{1 - y^2}$

$-1 \le y \le | \Rightarrow \dfrac{-1}{y} \ge 1$ of $\dfrac{-1}{y} \le -1$

$\therefore \text{Re}\left(\dfrac{2iz}{1 - z^2}\right) \in (-\infty, -1] \cup [1, \infty) \quad \therefore A \to s$

(B) For the domain of $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1 - 3^{2(x-1)}}\right)$

We should have

$$-1 \le \left(\frac{8(3)^{x-2}}{1 - 3^{2(x-1)}}\right) \le 1 \Rightarrow -1 \le \frac{8.3^x}{9 - 3^{2x}} \le 1$$

$$\Rightarrow \frac{8.3^x}{9 - 3^{2x}} \ge -1 \Rightarrow \frac{8.3^x + 9 - 3^{2x}}{9 - 3^{2x}} \ge 0$$

$$\Rightarrow \frac{(3^x-9)(3^x+1)}{(3^x-3)(3^x+3)} \ge 0$$

We know that $3^x > 0$

$$\begin{array}{ccccccc} & + & & - & & + & \\ -\infty & & 1 & & 2 & & \infty \end{array}$$

$\therefore\ x \in (-\infty, 1) \cup (2, \infty)$ ...(i)

And $\frac{8.3^x}{9-3^{2x}} \le 1 \Rightarrow \frac{8.3^x - 9 + 3^{2x}}{9-3^{2x}} \le 0$

$$\frac{(3^x+9)(3^x-1)}{(3^x-3)(3^x+3)} \ge 0$$

$$\begin{array}{ccccccc} & + & & - & & + & \\ -\infty & & 0 & & 1 & & \infty \end{array}$$

$\therefore x \in (-\infty, 0] \cup (1, \infty)$ ...(ii)

From (i) and (ii), we get $x \in (-\infty, 0] \cup [2, \infty)$ $\therefore$ B $\to$ t

(C) $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

Applying $R_1 = R_1 + R_3$

$$= \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$$

$= 2(1 + \tan^2\theta) = 2\sec^2\theta \ge 2$ for $0 \le \theta < \frac{\pi}{2}$ $\therefore$ C $\to$ r

(D) $f(x) = x^{3/2}(3x-10), x \ge 0$

$\therefore f'(x) = \frac{3}{2}x^{1/2}(3x-10) + x^{3/2}$

For f(x) to be increasing $f'(x) \ge 0$

$\Rightarrow 3x^{3/2}[3x - 10 + 2x] \ge 0 \Rightarrow x^{3/2}(5x-10) \ge 0$

$$\begin{array}{ccccccc} & - & & - & & + & \\ -\infty & & 0 & & 2 & & \infty \end{array}$$, But $x > 0$

$\therefore f(x)$ is incresing on $[2, \infty)$

$\therefore$ D $\to$ r.

32. **(A) $\to$ (r), (s), (p); (B) $\to$ (q), (s); (C) $\to$ (q), (s);** (D) $\to$ (r), (s), (p)

$$f(x) = \frac{x^2-6x+5}{x^2-5x+6} = \frac{(x-5)(x-1)}{(x-2)(x-3)}$$

(A) If $-1 < x < 1$ then $f(x) = \frac{(-ve)(-ve)}{(-ve)(-ve)} = +ve$

$\therefore$ $f(x) > 0$ (r)

Also $f(x) - 1 = \frac{-x-1}{x^2-5x+6} = -\frac{(x+1)}{(x-2)(x-3)}$

For $-1 < x < 1$, $f(x) - 1 = \frac{-(+ve)}{(-ve)(-ve)} = -ve$

$\Rightarrow f(x) - 1 < 0 \Rightarrow f(x) < 1$ (s)

$\therefore$ $0 < f(x) < 1$ (p)

(B) If $1 < x < 2$ then $f(x) = \frac{(-ve)(+ve)}{(-ve)(-ve)} = -ve$

$\therefore f(x) < 0$ (q) and so $f(x) < 1$ (s)

(C) If $3 < x < 5$ then $f(x) = \frac{(-ve)(+ve)}{(+ve)(+ve)} = -ve$

$\therefore f(x) < 0$ (q) and so $f(x) < 1$ (s)

(D) For $x > 5$, $f(x) > 0$ (r)

Also $f(x) - 1 = \frac{-(x+1)}{(x-2)(x-3)} < 0$

For $x > 5$, $f(x) < 1$ (s)

$\therefore 0 < f(x) < 1$ (p)

33. (A) $\to$ (q); (B) $\to$ (r)

(A) $f(x) = 1 + 2x, D_f = (-\pi/2, \pi/2)$

The given function represents a straight line so it is one one.

But $f_{min} = 1 - \pi = f\left(-\frac{\pi}{2}\right)$, $f_{max} = 1 + \pi = f\left(\frac{\pi}{2}\right)$

$\therefore$ Range of $f = (1-\pi, 1+\pi) \in (-\infty, \infty)$

$\therefore f$ is not onto. Hence (A) $\to$ (q).

(B) $f(x) = \tan x$

It is an increasing function on $(-\pi/2, \pi/2)$ and its range $= (-\infty, \infty)$ = co-domain of f.

$\therefore f$ is one one onto. Hence (B) $\to$ r

34. **(20)** Given $S = \{1, 2, 3, 4, 5, 6\}$ $R : S \to S$

Number of elements in $R = 6$

and for each $(a, b) \in R : |a - b| \ge 2$

$X \to$ set of all relation $R : S \to S$

$a = 1, b = 3, 4, 5, 6 \to$ (4)

$a = 2, b = 4, 5, 6 \to$ (3)

$a = 3, b = 1, 5, 6 \to$ (3)

$a = 4, b = 1, 2, 6 \to$ (3)

$a = 5, b = 1, 2, 3 \to$ (3)

$a = 6, b = 1, 2, 3, 4 \to$ (4)

Total number of ordered pairs (a, b) such that $|a - b| \ge 2 = 20$

$\therefore n(X) =$ number of elements in $X = {}^{20}C_6$

$\therefore m = 20$

35. **(36)** Given set $S = \{1, 2, 3, 4, 5, 6\}$; $R : S \to S$

Number of elements in $R = 6$

and for each $(a, b) \in R$; $|a - b| \ge 2$

$X \to$ set of all relation $R : S \to S$

If

$a = 1$	$b = 3, 4, 5, 6$	$\to$	4
$a = 2$	$b = 4, 5, 6$	$\to$	3
$a = 3$	$b = 1, 5, 6$	$\to$	3
$a = 4$	$b = 1, 2, 6$	$\to$	3
$a = 5$	$b = 1, 2, 3$	$\to$	3
$a = 6$	$b = 1, 2, 3, 4$	$\to$	4

Total number of ordered pairs (a, b) such that $|a - b| \ge 2 = 20$

$\therefore n(X) =$ number of elements in $X = {}^{20}C_6$

$\therefore m = 20$

$Y = \{R \in X$: The range of R has exactly one element)

From above, if range of R has exactly one element, then maximum number of elements in R will be 4.

$\therefore n(Y) = 0$

$Z = \{R \in X : R \text{ is a function from S to S}\}$

$n(Z) = {}^4C_1 \times {}^3C_1 \times {}^3C_1 \times {}^3C_1 \times {}^3C_1 \times {}^4C_1 = (36)^2$

$n(y) + n(z) = 0 + (36)^2 = k^2$

$\Rightarrow |k| = 36$

36. Let $f(x) = Ax^2 + Bx + C$ is an integer whenever x is an integer.

$\therefore$ $f(0), f(1), f(-1)$ are integers

$\Rightarrow$ $C, A + B + C, A - B + C$ are integers.

$\Rightarrow$ $C, A + B, A - B$ are integers

$\Rightarrow$ $C, A + B, (A + B) + (A - B) = 2A$ are integers.

Conversely suppose $2A$, $A + B$ and C are integers.

Let x be any integer.

Now, $f(x) = Ax^2 + Bx + C = 2A\left[\frac{x(x-1)}{2}\right] + (A + B)\ x + C$

Since x is an integer, therefore $x(x-1)/2$ will be also an integer.

Also $2A$, $A + B$ and C are integers.

$\therefore$ $f(x)$ is an integer for all integer x.

37. Put $y = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$

$\Rightarrow$ $(\alpha + 6x - 8x^2)\,y = \alpha x^2 + 6x - 8$

$\Rightarrow$ $(\alpha + 8y)\,x^2 + 6\,(1 - y)\,x - (8 + \alpha y) = 0$

$\because$ x is real, $\therefore$ $D \ge 0$

$\Rightarrow$ $36\,(1 - y)^2 + 4\,(\alpha + 8y)\,(8 + \alpha y) \ge 0$

$\Rightarrow$ $9\,(1 - 2y + y^2) + [8\alpha + (64 + \alpha^2)y + 8\alpha y^2] \ge 0$

$\Rightarrow$ $y^2\,(9 + 8\alpha) + y\,(46 + \alpha^2) + (9 + 8\alpha) \ge 0$ ...(i)

For (i) to hold for each $y \in R$,

$9 + 8\,\alpha > 0$ and $(46 + \alpha^2)^2 - 4\,(9 + 8\alpha)^2 \le 0$

$\Rightarrow \alpha > -9/8$ and $[46 + \alpha^2 - 2\,(9 + 8\alpha)]\,[46 + \alpha^2 + 2\,(9 + 8\alpha)] \le 0$

$\Rightarrow$ $\alpha > -9/8$ and $(\alpha^2 - 16\alpha + 28)\,(\alpha^2 + 16\alpha + 64) \le 0$

$\Rightarrow$ $\alpha > -9/8$ and $(\alpha - 2)\,(\alpha - 14)\,(\alpha + 8)^2 \le 0$

$\Rightarrow \alpha > -8/9$ and $(\alpha - 2)\,(\alpha - 14) \le 0$ $[\because (\alpha + 8)^2 \ge 0]$

$\Rightarrow \alpha > -8/9$ and $2 \le \alpha \le 14 \Rightarrow 2 \le \alpha \le 14$

Thus, $f(x) = \frac{\alpha x^2 + 6x - 8}{\alpha + 6x - 8x^2}$ will be onto if $2 \le \alpha \le 14$.

When $\alpha = 3$, then $f(x) = \frac{3x^2 + 6x - 8}{3 + 6x - 8x^2}$

In this case, $f(x) = 0$ implies, $3x^2 + 6x - 8 = 0$

$\Rightarrow$ $x = \frac{-6 \pm \sqrt{36 + 96}}{6} = \frac{-6 \pm 2\sqrt{33}}{6} = \frac{1}{3}(-3 \pm \sqrt{33})$

$\therefore$ $f\left[\frac{1}{3}(-3 + \sqrt{33})\right] = f\left[\frac{1}{3}(-3 - \sqrt{33})\right] = 0$

Hence, f is not one-to-one at $\alpha = 3$.

38. Given : $4\,\{x\} = x + [x]$,

where $[x]$ = greatest integer $\le x$

$\{x\}$ = fractional part of x

$\therefore$ $x = [x] + \{x\}$ for any $x \in R$

$\therefore$ Given equation becomes

$4\,\{x\} = [x] + \{x\} + [x]$ $\Rightarrow$ $3\,\{x\} = 2\,[x]$

$\Rightarrow$ $[x] = \frac{3}{2}\,\{x\}$ ...(i)

Now $-1 < \{x\} < 1 \Rightarrow -\frac{3}{2} < \frac{3}{2}\{x\} < \frac{3}{2}$

$\Rightarrow$ $-\frac{3}{2} < [x] < \frac{3}{2} \Rightarrow [x] = -1, 0, 1$ (using eqn (i))

If $[x] = -1$

$\Rightarrow -1 = \frac{3}{2}\{x\} \Rightarrow \{x\} = -\frac{2}{3}$ (using eqn (i))

$\therefore$ $x = [x] + \{x\} \Rightarrow x = -1 + (-2/3) = -5/3$

If $[x] = 0$, then $\frac{3}{2}\{x\} = 0$

$\Rightarrow$ $\{x\} = 0$ $\therefore$ $x = 0 + 0 = 0$

If $[x] = 1$, then $\frac{3}{2}\{x\} = 1$

$\Rightarrow$ $\{x\} = 2/3$ $\Rightarrow x = 1 + 2/3 = 5/3$

$\therefore$ $x = -5/3, 0, 5/3$

39. Given : $f(x + y) = f(x)f(y)\ \forall\ x, y \in N$ and $f(1) = 2$

To find 'a' such that $\sum_{k=1}^{n} f(a + k) = 16(2^n - 1)$ ...(i)

For this we start with $f(1) = 2$ ...(ii)

$\therefore f(2) = f(1 + 1) = f(1)\,f(1) \Rightarrow f(2) = 2^2$ [using (ii)]

Similarly we get, $f(3) = 2^3$, $f(4) = 2^4$, ..., $f(n) = 2^n$

Now eq. (i) can be written as

$f(a + 1) + f(a + 2) + f(a + 3) + \ldots + f(a + n) = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\,f(1) + f(a)\,f(2) + f(a)\,f(3) + \ldots + f(a)\,f(n) = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\,[f(1) + f(2) + f(3) + \ldots + f(n)] = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\,[2 + 2^2 + 2^3 + \ldots + 2^n] = 16\,(2^n - 1)$

$\Rightarrow$ $f(a)\left[\frac{2(2^n - 1)}{2 - 1}\right] = 16(2^n - 1)$

$\therefore$ $f(a) = 8 = 2^3 = f(3) \Rightarrow a = 3$

40. Since $|f(x) - f(y)| \le |x - y|^3$ is true $\forall\ x, y \in R$

For $x \ne y$, $\frac{|f(x) - f(y)|}{|x - y|} \le |x - y|^2$

$\Rightarrow$ $\lim_{y \to x}\left|\frac{f(x) - f(y)}{x - y}\right| \le \lim_{y \to x}|x - y|^2$

$\Rightarrow$ $\left|\lim_{y \to x}\frac{f(x) - f(y)}{x - y}\right| \le 0$

$\Rightarrow$ $|f'(x)| \le 0$ $\Rightarrow f'(x) = 0$

$\therefore$ $f(x)$ is a constant function.

41. Given that $z_1\,R\,z_2$ iff $\frac{z_1 - z_2}{z_1 + z_2}$ is real.

For reflexive :

$\because$ $\frac{z - z}{z + z} = 0$ which is real

$\therefore$ $z\,R\,z\ \forall\ z$ $\therefore$ R is reflexive.

For symmetric : Let $z_1\,R\,z_2 \Rightarrow \frac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow -\left(\frac{z_1 - z_2}{z_1 + z_2}\right)$ is also real

$\Rightarrow$ $\frac{z_2 - z_1}{z_2 + z_1}$ is real $\Rightarrow z_2\,R\,z_1$

$\therefore$ R is symmetric.

For transitive :

Let $z_1\,R\,z_2$

$\Rightarrow$ $\frac{z_1 - z_2}{z_1 + z_2}$ is real and $\frac{z_2 - z_3}{z_2 + z_3}$ is also real

$\Rightarrow$ $\frac{z_1 - z_2}{z_1 + z_2}$ is real $\Rightarrow$ $I_m\left(\frac{z_1 - z_2}{z_1 + z_2}\right) = 0$

$\Rightarrow \quad I_m\left(\frac{(x_1 - x_2) + i(y_1 - y_2)}{(x_1 + x_2) + i(y_1 + y_2)}\right) = 0$

$\Rightarrow \quad I_m\,((x_1 - x_2) + i\,(y_1 - y_2))\,((x_1 + x_2) - i\,(y_1 + y_2)) = 0$

$\Rightarrow \quad (x_1 + x_2)\ (y_1 - y_2) - (x_1 - x_2)\ (y_1 + y_2) = 0$

$\Rightarrow \quad x_2y_1 - x_1y_2 = 0 \Rightarrow \frac{x_1}{y_1} = \frac{x_2}{y_2}$ ...(i)

and $z_2\ R\ z_3$

Similarly, $I_m\left(\frac{z_2 - z_3}{z_2 + z_3}\right) = 0 \Rightarrow \frac{x_2}{y_2} = \frac{x_3}{y_3}$ ...(ii)

From (i) and (ii) we get $\frac{x_1}{y_1} = \frac{x_3}{y_3}$

$\Rightarrow \quad I_m\left(\frac{z_1 - z_3}{z_1 + z_3}\right) = 0 \Rightarrow \frac{z_1 - z_3}{z_1 + z_3}$ is real

$\Rightarrow \quad z_1 R\ z_3\ \therefore\ R$ is transitive.

Thus R is reflexive, symmetric and transitive.

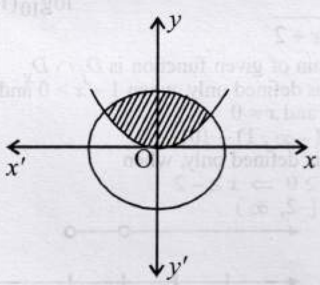
Hence R is an equivalence relation.

42. As there is an injective maping from A to B, each element of A has unique image in B. Similarly as there is an injective mapping from B to A, each element of B has unique image in A. So we can conclude that each element of A has unique image in B and each element of B has unique image in A or in other words there is one to one mapping from A to B. Thus there is bijective mapping from A to B.

43. $R = [(x, y); x \in R, y \in R, x^2 + y^2 \leq 25]$, which represents all the points inside and on the circle $x^2 + y^2 = 5^2$, with centre (0, 0) and radius = 5,

$R' = \left\{(x, y) : x \in R, y \in R, y \geq \frac{4}{9}x^2\right\}$,

which represents all the points inside and on the upward parabola $x^2 \leq \frac{9}{4}y$.



$\therefore \quad R \cap R'$ = The set of all points in shaded region.

Now, $x^2 + y^2 \leq 25 \Rightarrow x^2 \leq 25 - y^2$ ....(i)

and $y \geq \frac{4}{9}x^2 \Rightarrow \frac{16x^4}{81} \leq y^2$

$\Rightarrow \quad -\frac{16x^4}{81} \geq -y^2$

$\Rightarrow \quad 25 - \frac{16x^4}{81} \geq 25 - y^2$ ....(ii)

From (i) and (ii), $x^2 \leq 25 - \frac{16}{81}x^4$

$\Rightarrow \quad 16x^4 + 81x^2 - 2025 \leq 0$

$\therefore \quad$ Domain of $R \cap R'$ =

$\{x : x \in R, 16x^4 + 81x^2 - 2025 \leq 0\}$ and range of $R \cap R'$

$= \{y : y \in R, y \geq \frac{4x^2}{9}, 16x^4 + 81x^2 - 2025 \leq 0\}$

$R \cap R'$ is not a function because image of an element is not unique , e..g., (0, 1), (0, 2), (0,3) ..... $\in R \cap R'$.

44. $f(x) = x^9 - 6x^8 - 2x^7 + 12x^6 + x^4 - 7x^3 + 6x^2 + x - 3$

$\therefore\ f(6) = 6^9 - 6 \times 6^8 - 2 \times 6^7 + 12 \times 6^6 + 6^4 - 7 \times 6^3 + 6 \times 6^2 + 6 - 3$

$= 6^9 - 6^9 - 2 \times 6^7 + 2 \times 6^7 + 6^4 - 7 \times 6^3 + 6^3 + 6 - 3 = 3$

45. $y = |x|^{1/2}, -1 \leq x \leq 1$

$\Rightarrow \quad y = \sqrt{-x}$ if $-1 \leq x \leq 0 = \sqrt{x}$ if $0 \leq x \leq 1$

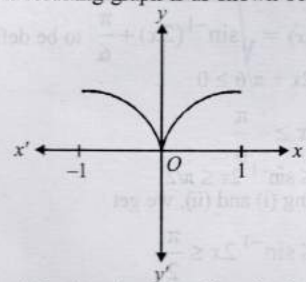
$\Rightarrow \quad y^2 = -x$ if $-1 \leq x \leq 0$ and $y^2 = x$ if $0 \leq x \leq 1$

[Here y should be taken always + ve, as by definition y is a + ve square root].

Clearly $y^2 = -x$ represents upper half of left handed parabola (upper half as y is + ve)

and $y^2 = x$ represents upper half of right handed parabola.

Therefore the resulting graph is as shown below :



46. Since $f(x)$ is defined and real for all real values of x,

$\therefore \quad$ Domain of f is R.

Clearly $0 \leq \frac{x^2}{1 + x^2} < 1$, for all $x \in R \Rightarrow 0 \leq f(x) < 1$

$\Rightarrow$ Range of $f = [0, 1)$

Since $f(1) = f(-1) = 1/2$

$\therefore \quad f$ is not one-to-one.

Topic-2: Composite Functions & Relations, Inverse of a Function, Binary Operations

1. **(a)** Given : $f(x) = x^2$ and $g(x) = \sin x,\ \forall\ x \in R$

$\therefore$ (gof) $(x) = \sin x^2$

$\Rightarrow$ (gogof) $(x) = \sin(\sin x^2)$

$\Rightarrow$ (fogogof) $(x) = \sin^2(\sin x^2)$

Since given that (fogogof) (x) = (gogof) (x)

$\therefore\ \sin^2(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow \sin(\sin x^2) = 0, 1$

$\Rightarrow \sin x^2 = n\pi$ or $\left((4n+1)\frac{\pi}{2}\right)$, where $n \in Z$

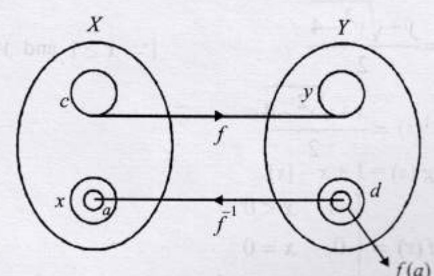
$\Rightarrow \sin x^2 = 0 \quad (\because \sin x^2 \in [-1, 1]) \Rightarrow x^2 = n\pi$

$\therefore\ x = \pm\sqrt{n\pi}$, where $n \in W$

2. **(d)** Given that X and Y are two sets and $f : X \to Y$.

$\{f(c) = y; c \subset X, y \subset Y\}$ and $\{f^{-1}(d) = x : d \subset Y, x \subset X\}$

The pictorial representation of given information is as shown:



Since $f^{-1}(d) = x \Rightarrow f(x) = d$.

Now if $a \subset x \Rightarrow f(a) \subset f(x) = d \Rightarrow f^{-1}(f(a)) = a$

Hence, $f^{-1}(f(a)) = a,\ a \subset x$ is the correct option.

3. (b) Given : $f(x) = \sin x + \cos x$ and $g(x) = x^2 - 1$
$\Rightarrow g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly $g(f(x))$ is invertible in $-\frac{\pi}{2} \le 2x \le \frac{\pi}{2}$
[$\because$ $\sin\theta$ is invertible in $-\pi/2 \le \theta \le \pi/2$]

$\Rightarrow -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

4. (d) $f(x) = x^2 + 2bx + 2c^2 \Rightarrow f(x) = (x+b)^2 + 2c^2 - b^2$
$\Rightarrow f_{min} = 2c^2 - b^2$
and $g(x) = -x^2 - 2cx + b^2$
$g(x) = -(x+c)^2 + b^2 + c^2 \Rightarrow g_{max} = b^2 + c^2$
For $f_{min} > g_{max} \Rightarrow 2c^2 - b^2 > b^2 + c^2$
$\Rightarrow c^2 > 2b^2 \Rightarrow |c| > |b|\sqrt{2}$

5. (a) For $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ to be defined and real,
$\sin^{-1} 2x + \pi/6 \ge 0$

$\Rightarrow \sin^{-1} 2x \ge -\frac{\pi}{6}$ ...(i)

But $-\pi/2 \le \sin^{-1} 2x \le \pi/2$ ...(ii)
On combining (i) and (ii), we get

$-\frac{\pi}{6} \le \sin^{-1} 2x \le \frac{\pi}{2}$

$\Rightarrow \sin(-\pi/6) \le 2x \le \sin(\pi/2) \Rightarrow -1/2 \le 2x \le 1$

$\Rightarrow -1/4 \le x \le 1/2$, $\therefore$ Domain $= \left[-\frac{1}{4}, \frac{1}{2}\right]$

6. (d) $f(x) = \frac{\alpha x}{x+1}, x \ne -1$

Now, $f(f(x)) = x \Rightarrow \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1} + 1} = x$

$\Rightarrow \frac{\alpha^2 x}{(\alpha+1)x+1} = x \Rightarrow (\alpha+1)x^2 + (1-\alpha^2)x = 0$
$\Rightarrow \alpha + 1 = 0$ and $1 - \alpha^2 = 0 \Rightarrow \alpha = -1$

7. (d) For domain of $f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$
$x^2 + 3x + 2 \ne 0$ and $x + 3 > 0$
$\Rightarrow x \ne -1, -2$ and $x > -3$
$\therefore$ Domain of $f(x) = (-3, \infty) - \{-1, -2\}$

8. (a) Given : $f(x) = x + \frac{1}{x} = y$ (let) $\Rightarrow x^2 - yx + 1 = 0$

$\Rightarrow x = \frac{y \pm \sqrt{y^2-4}}{2}$

$\therefore x = \frac{y + \sqrt{y^2-4}}{2}$ [$\because$ $x \ge 1$ and $y \ge 2$]

$\therefore f^{-1}(x) = \frac{x + \sqrt{x^2-4}}{2}$

9. (b) $g(x) = 1 + x - [x]$

and $f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

For integral values of x; $g(x) = 1$
For $x < 0$ (but not integral value); $x - [x] > 0 \Rightarrow g(x) > 1$
For $x > 0$ (but not integral value); $x - [x] > 0 \Rightarrow g(x) > 1$
$\therefore g(x) \ge 1, \forall x \Rightarrow f(g(x)) = 1, \forall x$

10. (b) Let $y = 2^{x(x-1)}$
$\Rightarrow x^2 - x - \log_2 y = 0;$

$x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\log_2 y}\right)$

For $y \ge 1$, $\log_2 y \ge 0 \Rightarrow \sqrt{1 + 4\log_2 y} \ge 0$

$\because x \ge 1$, $\therefore x = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 y}\right)$

$\Rightarrow f^{-1}(x) = \frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$

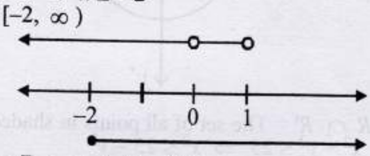
11. (c) $f(x) = f^{-1}(x) \Rightarrow fof(x) = x$
$\Rightarrow [(x+1)^2 - 1 + 1]^2 - 1 = x \Rightarrow (x+1)^4 = x + 1$
$\Rightarrow (x+1)[(x+1)^3 - 1] = 0$
$\therefore x = 0$ or -1
$\therefore$ Required set is $\{0, -1\}$

12. (d) Given : $f(x) = \sin x$ and $g(x) = \ell n\,|x|$
Now $f\,o\,g(x) = f(g(x)) = \sin(\ln|x|)$
$\therefore R_1 = \{u : -1 \le u \le 1\}$ ($\because -1 \le \sin\theta \le 1, \forall \theta$)
Also $g\,o\,f(x) = g(f(x)) = \ln|\sin x|$
$\because 0 \le |\sin x| \le 1$
$\therefore -\infty < \ln|\sin x| \le 0$
$\therefore R_2 = \{v : -\infty < v \le 0\}$

13. (c) $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$

$\Rightarrow y = f(x) + g(x)$, where $f(x) = \frac{1}{\log_{10}(1-x)}$ and

$g(x) = \sqrt{x+2}$
$\therefore$ Domain of given function is $D_f \cap D_g$
Since $f(x)$ is defined only, when $1 - x > 0$ and $1 - x \ne 1$
$\Rightarrow x < 1$ and $x \ne 0$
$\therefore D_f = (-\infty, 1) - \{0\}$
Also $g(x)$ is defined only, when
$x + 2 \ge 0 \Rightarrow x \ge -2$
$\therefore D_g = [-2, \infty)$



$\therefore D_f \cap D_g = [-2, 1) - \{0\}$

14. (d) Given : $f(x) = \cos(\ell n\,x)$

$\therefore f(x)f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$

$= \cos(\ell n\,x)\cos(\ell n\,y) - \frac{1}{2}[\cos(\ell n\,x - \ell n\,y) + \cos(\ell n\,x + \ell n\,y)]$

$= \cos(\ell n\,x)\cos(\ell n\,y) - \frac{1}{2}[2\cos(\ell n\,x)\cos(\ell n\,y)] = 0$

15. (8) $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$

$= (\log_2 9)^{2 \times \log_{(\log_2 9)} 2} \times 7^{\frac{1}{2} \times \log_7 4}$

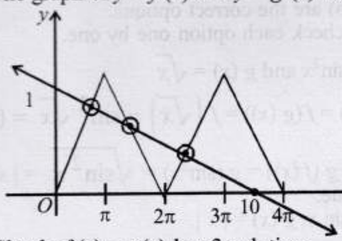
$= (\log_2 9)^{\log_{(\log_2 9)} 4} \times 7^{\log_7 2} = 4 \times 2 = 8$

16. **(3)** Given : $f:[0,4\pi]\to[0,\pi]$ defined by

$f(x)=\cos^{-1}(\cos x)$

and $g(x)=\dfrac{10-x}{10}=1-\dfrac{x}{10}$

The graph of $y=f(x)$ and $y=g(x)$ are as follows.



Clearly $f(x)=g(x)$ has 3 solutions.

17. $f(x)=\sin^2 x+\sin^2\left(x+\dfrac{\pi}{3}\right)+\cos x\cos\left(x+\dfrac{\pi}{3}\right)$

$$=\sin^2 x+\left[\sin\left(x+\frac{\pi}{3}\right)\right]^2+\cos x\cos\left(x+\frac{\pi}{3}\right)$$

$$=\sin^2 x+\frac{1}{4}(\sin x+\sqrt{3}\cos x)^2$$

$$+\frac{1}{2}\cos x(\cos x-\sqrt{3}\sin x)$$

$$=\frac{5}{4}(\sin^2 x+\cos^2 x)=\frac{5}{4}$$

$\therefore$ $(gof)x=g[f(x)]=g(5/4)=1$

18. Given function is, $f(x)=\sin\left[\ell n\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)\right]$

For $\ell n\left(\dfrac{\sqrt{4-x^2}}{1-x}\right)$ to be defined $\dfrac{\sqrt{4-x^2}}{1-x}>0$

$\Rightarrow$ $1-x>0$ and $4-x^2>0\Rightarrow x<1$ and $-2<x<2$

Combining these two inequalities, we get $x\in(-2,1)$

$\therefore$ Domain of $f(x)$ is $(-2,1)$

Since $\sin\theta$ always lies in $[-1,1]$.

$\therefore$ Range of $f(x)$ is $[-1,1]$

19. The function $f(x)=\sin^{-1}\left(\log_2\dfrac{x^2}{2}\right)$ will be defined

if $-1\le\log_2\left(\dfrac{x^2}{2}\right)\le 1$

$\Rightarrow$ $2^{-1}\le\dfrac{x^2}{2}\le 2^1$ $\Rightarrow 1\le x^2\le 4$

$\Rightarrow -2\le x\le -1$ or $1\le x\le 2\Rightarrow x\in[-2,-1]\cup[1,2]$

20. Given : $f(x)=3\sin\left(\sqrt{\dfrac{\pi^2}{16}-x^2}\right)$

For the given function to be defined

$\dfrac{\pi^2}{16}-x^2\ge 0\Rightarrow -\pi/4\le x\le\pi/4$

$\therefore$ Domain $=[-\pi/4,\pi/4]$

Now, for $x\in[-\pi/4,\pi/4]$, $\sqrt{\pi^2/16-x^2}\in[0,\pi/4]$

and sine function increases on $[0,\pi/4]$.

$\therefore$ $0\le\sin\sqrt{\dfrac{\pi^2}{16}-x^2}\le 1/\sqrt{2}$

$\Rightarrow$ $0\le 3\sin\sqrt{\dfrac{\pi^2}{16}-x^2}\le 3/\sqrt{2}$

$\therefore$ $f(x)=[0,3/\sqrt{2}]$

21. **(True)** $f(x)=(a-x^n)^{1/n}$, $a>0$, n is a positive integer

$f(f(x))=f[(a-x^n)^{1/n}]=[a-\{(a-x^n)^{1/n}\}^n]^{1/n}$

$=(a-a+x^n)^{1/n}=x$

22. **(a, b, c)**

$$f(x)=\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$$

$-1\le\sin x\le 1\Rightarrow -\dfrac{\pi}{2}\le\dfrac{\pi}{2}\sin x\le\dfrac{\pi}{2}$

$\Rightarrow$ $-1\le\sin\left(\dfrac{\pi}{2}\sin x\right)\le 1\Rightarrow\dfrac{-\pi}{6}\le\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\le\dfrac{\pi}{6}$

$\Rightarrow$ $\dfrac{-1}{2}\le\sin\left[\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right]\le\dfrac{1}{2}$

$\therefore$ Range of $f=\left[\dfrac{-1}{2},\dfrac{1}{2}\right]$

Now, $fog(x)=\sin\left[\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)\right]$

Range of $fog=\left[\dfrac{-1}{2},\dfrac{1}{2}\right]$

Now, $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 0}\dfrac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{2}\sin x}$

$$=\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)}{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}\times\frac{\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x}=\pi/6$$

$$gof(x)=\frac{\pi}{2}\sin\left(\sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)\right)$$

$$-\frac{\pi}{2}\sin\left(\frac{1}{2}\right)\le g(f(x))\le\frac{\pi}{2}\sin\left(\frac{1}{2}\right)$$

Let $\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)=p$

Clearly $0<p<1$

$\therefore$ $-\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)\le g(f(x))\le\dfrac{\pi}{2}\sin\left(\dfrac{1}{2}\right)$

$-p\le g(f(x))\le p\Rightarrow o<p<1$

$\therefore$ $gof(x)\ne 1$ for any $x\in R$.

23. **(a, b, c)** Given : $f:\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)\to R$ is given by

$f(x)=(\log(\sec x+\tan x))^3$

$f(-x) = (\log(\sec x - \tan x))^3$

$$= \left[\log\left(\frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x}\right)\right]^3$$

$$= \left[\log\left(\frac{1}{\sec x + \tan x}\right)\right]^3 = \left[-\log(\sec x + \tan x)\right]^3$$

$$= -\left[\log(\sec x + \tan x)\right]^3 = -f(x)$$

$\therefore$ $f(x)$ is an odd function.

$\therefore$ option (a) is correct and (d) is not correct.

Now, $f'(x) = 3\left[\log(\sec x + \tan x)\right]^2 \cdot \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$= 3\sec x\left[\log(\sec x + \tan x)\right]^2 > 0 \ \forall x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore$ $f(x)$ is increasing on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

We know that strictly increasing function is one one.

$\therefore f$ is one one, hence (b) is the correct option.

Also $\lim\limits_{x \to \frac{\pi}{2}^-} \left[\log(\sec x + \tan x)\right]^3 \to \infty$

and $\lim\limits_{x \to -\frac{\pi}{2}^+} \left[\log(\sec x + \tan x)\right]^3 \to -\infty$

$\therefore$ Range of $f = (-\infty, \infty) = R =$ Domain

$\therefore f$ is an onto function.

$\therefore$ option (c) is correct.

24. **(a, b)** Given : $f(x) = \dfrac{b-x}{1-bx}, 0 < b < 1$

Let $f(x_1) = f(x_2) \Rightarrow \dfrac{b-x_1}{1-bx_1} = \dfrac{b-x_2}{1-bx_2}$

$\Rightarrow b - b^2 x_2 - x_1 + bx_1x_2 = b - x_2 - b^2x_1 + bx_1 x_2$

$\Rightarrow x_2(1-b^2) = x_1(1-b^2) \Rightarrow x_1 = x_2$ as $1 - b^2 \neq 0$

$\therefore f$ is one one.

Also $\dfrac{b-x}{1-bx} = y \Rightarrow b - x = y - bxy$

$\Rightarrow (by - 1)x = y - b \Rightarrow x = \dfrac{y-b}{by-1}$

For $y = \dfrac{1}{b}$, x is not defined

$\therefore f$ is not onto and hence nor invertible.

Also $f'(x) = \dfrac{-1(1-bx)-(-b)(b-x)}{(1-bx)^2} = \dfrac{b^2-1}{(1-bx)^2}$

$\therefore \quad f'(b) = \dfrac{1}{b^2-1}$ and $f'(0) = b^2 - 1 \Rightarrow f'(b) = \dfrac{1}{f'(0)}$

$\therefore$ (a) and (b) are the correct options.

25. **(a)** Let us check each option one by one.

(a) $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Now, $fog(x) = f(g(x)) = f\left(\sqrt{x}\right) = \sin^2\sqrt{x} = (\sin\sqrt{x})^2$

and $gof(x) = g(f(x)) = g(\sin^2 x) = \sqrt{\sin^2 x} = |\sin x|$

$\therefore$ (a) is true.

(b) $f(x) = \sin x$, $g(x) = |x|$

$fog(x) = f(g(x)) = f(|x|) = \sin|x| \neq \left(\sin\sqrt{x}\right)^2$

$\therefore$ (b) is not true

(c) $f(x) = x^2$, $g(x) = \sin\sqrt{x}$

$fog(x) = f(g(x)) = f\left(\sin\sqrt{x}\right) = \left(\sin\sqrt{x}\right)^2$

and $(gof)(x) = g(f(x)) = g(x^2) = \sin\sqrt{x^2}$
$= \sin|x| \neq |\sin x|$

$\therefore$ (c) is not true.

26. **(b)** $f(x) = 3x - 5$ is strictly increasing on R.

$\therefore f^{-1}(x)$ exists.

Let $y = f(x) = 3x - 5$

$\Rightarrow y + 5 = 3x \Rightarrow x = \dfrac{y+5}{3}$ ...(i)

$\because y = f(x) \Rightarrow x = f^{-1}(y)$ ...(ii)

From (i) and (ii),

$f^{-1}(y) = \dfrac{y+5}{3} \Rightarrow f^{-1}(x) = \dfrac{x+5}{3}$

27. **(a)** For E_1, $\dfrac{x}{x-1} > 0$ and $x \neq 1 \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

For E_2, $-1 \le \log_e\left(\dfrac{x}{x-1}\right) \le 1 \Rightarrow \dfrac{1}{e} \le \dfrac{x}{x-1} \le e$

$\Rightarrow \dfrac{1}{e} \le 1 + \dfrac{1}{x-1} \le e \Rightarrow \dfrac{1}{e} - 1 \le \dfrac{1}{x-1} \le e - 1$

$\Rightarrow (x-1) \in \left(-\infty, \dfrac{e}{1-e}\right] \cup \left[\dfrac{1}{e-1}, \infty\right)$

$\Rightarrow x \in \left(-\infty, \dfrac{1}{e-1}\right] \cup \left[\dfrac{e}{e-1}, \infty\right)$

For E_1, $\dfrac{x}{x-1} \in (0, \infty) - \{1\}$

$\Rightarrow \log_e\left(\dfrac{x}{x-1}\right) \in (-\infty, \infty) - \{0\}$

$\Rightarrow f(x) \in (-\infty, 0) \cup (0, \infty)$

$g(x) = \sin^{-1}\left(\log_e\left(\dfrac{x}{x-1}\right)\right) \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$

28. f is one one function,

$D_f = \{x, y, z\}$; $R_f = \{1, 2, 3\}$

Exactly one of the following is true :

$f(x) = 1, f(y) \neq 1, f(z) \neq 2$

To determine f^{-1} (1):

Case I: $f(x) = 1$ is true.

$\Rightarrow$ $f(y) \neq 1, f(z) \neq 2$ are false.

$\Rightarrow$ $f(y) = 1, f(z) = 2$ are true.

But $f(x) = 1, f(y) = 1$ are true, is not possible as f is one to one.

$\therefore$ This case is not possible.

Case II: $f(y) \neq 1$ is true.

$\Rightarrow$ $f(x) = 1$ and $f(z) \neq 2$ are false

$\Rightarrow$ $f(x) \neq 1$ and $f(z) = 2$ are true

Thus, $f(x) \neq 1, f(y) \neq 1, f(z) = 2$

$\Rightarrow$ Either $f(x)$ or $f(y) = 2$. So, f is not one to one

$\therefore$ This case is also not possible.

Case III: $f(z) \neq 2$ is true

$\therefore$ $f(x) = 1$ and $f(y) \neq 1$ are false.

$\Rightarrow$ $f(x) \neq 1$ and $f(y) = 1$ are true.

$\therefore$ $f^{-1}(1) = y$